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Renormalizing the Lippmann-Schwinger equation for the one-pion exchange potential

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Abstract. We address the question whether the cut-off dependence, which has to be introduced in order to properly define the Lippmann-Schwinger equation for the one-pion exchange potential plus local (δ-function) potentials, can be removed (up to inverse powers of it) by a suitable tuning of the various (bare) coupling constants. We prove that this is indeed so both for the spin singlet and for the spin triplet channels. However, the latter requires, in the limit when the cut-off is taken to infinity, such a strong cut-off dependence of the coupling constant associated to the non-local term which breaks orbital angular momentum conservation, that the renormalized amplitude lacks from partial-wave mixing. We argue that this is an indication that this term must be treated perturbatively.

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1 Introduction

Since the original suggestion by Weinberg [1] that the nuclear forces could be understood within the framework of effective field theories (EFT), there has been an increasing interest in the subject (see [2] for recent reviews). A key ingredient of the EFT formalism is that the cut-off dependence which is introduced in order to smooth out ultraviolet (UV) singularities can be absorbed by suitable counterterms, and hence any dependence on physical scales much higher than the ones of the problem at hand can be encoded in a few (unknown) constants. In order to achieve this in a systematic manner, counting rules are also necessary.

Weinberg's suggestion consisted of two steps. The first one was calculating the nucleon-nucleon (NN) potentials order by order in Chiral Perturbation Theory (χPT) from the Heavy Baryon Chiral Lagrangian (HBχL) [3]. The second one introducing the potentials thus obtained in a Lippmann-Schwinger (LS) equation. There is no doubt that the first step can be carried out within an EFT framework: the renormalized NN potentials are known at leading, NL, NNL [4,5] and NNNL order [6], and isospin breaking terms have also been taken care of [7]. The second step however is delicate. The potentials obtained in the first step are increasingly singular at short distances as we rise the order of χPT they are calculated. Hence the introduction of a regulator in the LS equation is compulsory. Since, even with the leading-order (LO) potential, the LS equation can only be solved numerically, it is not clear that the scattering amplitude thus obtained is cut-off independent. This is so even for the successful fits [4,5] to different partial amplitudes, where the cut-off is regarded as a variational parameter close to the last scale integrated out. We present here a proof that this cut-off can be removed from the LO (in the χPT counting) NN interaction if we tune properly the coupling constants of the potential. However, for this to be so we also have to tune the coupling constant of a non-local potential in the triplet channel. Even then, the only solution we find turns out to be physically unacceptable. Nevertheless, the insight on scaling so gained enables us to put forward a new proposal of counting rules where, coming back to standard procedures, divergences are fully absorbed by local counterterms.

As EFTs have been mainly used in a perturbative framework, it is far from obvious how the two main features of them, namely renormalizability and counting rules, must be implemented in a non-perturbative one. Although in this work we shall primarily address the question of renormalizability, we would like to start by making a remark on counting rules, which emanates from previous experience on EFT in non-perturbative systems. It was pointed out in ref. [8] that calculating the potential

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in a non-relativistic system can be understood as the integration of certain degrees of freedom, which can be implemented as a matching calculation between two EFTs. In our case, the higher-energy EFT is the HBχL for the two-nucleon sector, which is a local theory with pions and non-relativistic nucleons as explicit degrees of freedom. This EFT has an energy (E) cut-off (Λ_E) such that $E \sim m_\pi \ll \Lambda_E \ll M \sim 4\pi f_\pi$ and a momentum (p) cut-off (Λ_p) such that $p \ll \Lambda_p \ll M \sim 4\pi f_\pi$ (m_π and M stand for the pion and nucleon mass, respectively, and $4\pi f_\pi$ for the scale of non-Goldstone boson QCD states). Its Lagrangian can be organized according to the chiral counting since chiral symmetry (and its breaking) are explicit. The lower-energy EFT has an energy cut-off (Λ'_E) such that $E \ll \Lambda'_E \ll m_\pi$ and a momentum cut-off such that $p \lesssim m_\pi \ll \Lambda'_p \ll M$. It consists of non-relativistic nucleons interacting through a (non-local) potential. The potential plays the role of a matching coefficient. As such, the potential encodes information on the higher-energy EFT and can be calculated *independently* of how the calculation of the lower-energy EFT is organized, namely *independently* of what the counting in the low-energy EFT is. Hence, on the one hand, the potential can be calculated order by order in χPT, for instance along the lines of ref. [5]. On the other hand, chiral symmetry is not explicit anymore in the lower-energy EFT (no pion fields exist) and, consequently, the chiral counting is not the natural way to organize the calculation anymore[1].

An interesting example of a related situation is the pionium system (see [9] for a recent account), which has been studied using a series of EFTs [10]. The higher-energy EFT is the Chiral Lagrangian coupled to electromagnetism and the lower-energy one a quantum-mechanical Hamiltonian with the Coulomb potential and local interactions. The matching between the two EFTs can be carried out perturbatively in χPT and α, but the calculations in the lower EFT are carried out keeping the Coulomb potential non-perturbatively (otherwise no bound state exists) and, furthermore, one does not need to specify to which order of χPT the local potentials have been calculated.

Following that spirit, the main question for the NN system is what should be treated as the LO potential in the low-energy calculations. In the (higher energy) χPT counting the LO potential consists of the one-pion exchange term (OPE) plus two local (δ-function) terms. This assumes that the natural scale of the two local terms is of the order of the last scale integrated out ($\sim M$). If the NN system were in a perturbative regime the scale of these two local terms would provide the scale of the scattering lengths. Since the experimental scattering lengths are much larger than the ones predicted in this way, we can foresee at least two possibilities. The first one is that an unsuspected behavior of QCD at energies $\sim \Lambda_{\rm QCD}$ produces unnaturally large values for the local potentials. Then one may consider these local terms as the (low-energy) LO potential and treat the OPE (and higher orders) perturbatively [11]. This approach has been worked out at N^{2}LO [12] showing slow convergence in the $^1S_0$ channel and no convergence at all in the $^3S_1$-$^3D_1$ channel. The second possibility is that the local terms do have natural sizes but the low-energy dynamics is responsible for the large scattering lengths. In this case there is no reason to treat the OPE perturbatively and a fully non-perturbative evaluation of the LS equation with LO potential (in the χPT counting) is required [4,5]. We shall stick to this second possibility for most of the paper, although eventually a third possibility, which is half-way, will emerge as the most reasonable one (to us).

Before going on, let us briefly discuss some previous work on the renormalization of the LS equation. The case of a local (and hence separable) potential, namely consisting of delta-functions and its derivatives, has received plenty of attention [13–15]. This was expected to mimic the very low-energy ($p \ll m_\pi$) behavior of NN scattering. The regularization of this pure local EFT was a matter of debate some time ago: a cut-off regularization showed a systematic order-by-order improvement in the phase shift fit, whereas dimensional regularization (DR) with MS scheme was extremely sensible to the large scattering length and shallow (nearly)-bound state, which translated into a poor radius of convergence. The shortcomings of DR with MS were cured using the PDS scheme [11] (see also [16]). The final outcome appears to be equivalent to the well-known Effective Range Expansion [17]. The next step in difficulty is renormalizing the LO potential in the $^1S_0$ channel, which contains a non-separable piece from the OPE. It was first carried out in [18], and reproduced by several authors (see [19], for a recent report). We shall reobtain these results in sect. 3. Finally, as for renormalization in the $^3S_1$-$^3D_1$ channel, the available literature is, on the contrary, somewhat scarce [19,20] and the results are, to our understanding, not fully satisfactory (see sect. 6).

The main difference of our approach with respect to the previous ones is that, in addition to the bare constants associated to local terms in the potential, we will also allow, but only in an initial stage, the bare constants of the non-local potentials to have non-trivial flows. This possibility, which was already mentioned (but not developed) in ref. [21], is less restrictive than the standard assumption that only local terms should renormalize the LS equation for NN systems[2], which, in any case, is contained in it.

[1] Note that the range of energies for which this EFT holds excludes relativistic pions as explicit degrees of freedom. Hence, processes involving relativistic pions, like pion-deuteron scattering at pion three-momenta of the order of m_π, cannot be directly computed in it. One has first to separate the two-nucleon low-energy sub-process of the whole scattering process and then apply the EFT to this subprocess only. On the other hand, non-relativistic pions (*i.e.* pions with three-momenta much smaller than m_π) could be easily incorporated in the EFT.

[2] In fact, this turns out to be the usual approach in theoretical works on renormalization of singular potentials (see, for instance, [22]). Besides, there are known examples in a non-relativistic EFT of QCD (pNRQCD) where the renormalization of non-local potentials is required in order to absorb certain divergences [23], the most spectacular of which being the renormalization of the static potential [24].

This will allow us, not only to make meaningful comparisons with related work, but also to draw restrictions on the power counting. Having examined which conditions on the coupling constants are required in order to renormalize the LS equation, and the eventual consequences this has on our observables, we will be able to glean which terms of the potential can be included at LO, and which ones must be treated as perturbations, in the relevant case where only the coupling constants of local terms are allowed to flow and non-local potentials are fixed at the HBχPT values.

So, once the (low-energy) LO potential has been identified, we suspect that, in order to be renormalizable, a higher-order calculation should be organized as follows. The LS equation must be solved and renormalized treating the LO potential, as well as its couplings, non-perturbatively, but the NLO potentials and higher perturbatively.

Therefore, the first step in this program is to identify a (low-energy) LO potential and to prove that it is renormalizable. We start by the naive choice, namely we take it with the form of the LO potential in the χPT counting. For the spin singlet channel the LS equation is indeed renormalizable[3]. However, for the spin triplet channels, we find that a (low-energy) LO potential with the form of the LO potential in the χPT counting is only renormalizable if a certain coupling constant of a non-local potential has a non-trivial flow. (Or, in other words, if only the coupling constants of the local potentials are allowed to flow it is non-renormalizable.) Even in this case, the physical outcome is not satisfactory: the partial-wave mixing is washed out of the renormalized amplitude. We conclude that the (low-energy) LO potential must not have the form of the full LO potential in the χPT counting for the triplet channels. We identify a (low-energy) LO potential, which is renormalizable, and prove that, if we treat the difference as a perturbation, it is also renormalizable at first order.

We distribute the paper as follows. In sect. 2 we introduce a convenient basis for the NN wave functions and our notations. A brief note at the end of this section serves to close all what refers to the isosinglet-singlet channel. In sect. 3 we prove that the isovector-singlet channel is renormalizable and provide explicit expressions for the cut-off dependence of the bare parameters both for a hard cut-off and for dimensional regularization. In sect. 4 we prove that the isosinglet-triplet channel is also renormalizable, but requires a strong cut-off dependence of the coupling constant of the (non-local) term in the potential which, in turn, prevents the renormalized amplitude from partial-wave mixing. We interpret this result as an indication that this term must be treated perturbatively and prove that, if so, the first order in perturbation theory is finite. After briefly discussing in sect. 5 the isovector-spin vector channel, sect. 6 is devoted to a discussion. Appendices A and B contain technical details. Appendix C explores the possibility of having a non-trivial fixed point. Appendix D displays technical details which are relevant for comparing our results with those of ref. [19].

[3] The counterterm needed is NLO in the χPT counting, which in our approach only means that the matching between HBχL and our EFT must be done for consistency at NLO for the local terms.

2 A convenient decomposition

We start from a potential with the form of the LO NN potential in the χPT counting, given, for instance, in ref. [5]:

$$V(\mathbf{k},\mathbf{k}') = -\left(\frac{g_A}{2f_\pi}\right)^2 \tau_1\cdot\tau_2 \frac{\sigma_1\cdot(\mathbf{k}-\mathbf{k}')\,\sigma_2\cdot(\mathbf{k}-\mathbf{k}')}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + C_S + C_T\,\sigma_1\cdot\sigma_2\,, \tag{2.1}$$

where f_π is the pion decay constant (~ 93 MeV).

This potential acts on a wave function $\Psi^{ab}_{\alpha\beta}(\mathbf{k})$, where a,b and α,β are nucleon isospin and spin indices, respectively. This wave function can be decomposed into irreducible representations of spin and isospin as follows:

$$\begin{aligned}\Psi^{ab}_{\alpha\beta}(\mathbf{k}) = \frac{1}{2}\Big[&(\tau_2)^{ab}(\sigma_2)_{\alpha\beta}\,\psi_{SS}(\mathbf{k}) \\ &+(\tau_2)^{ab}(\sigma_{k'}\sigma_2)_{\alpha\beta}\,\psi^{k'}_{SV}(\mathbf{k}) \\ &+(\tau_k\tau_2)^{ab}(\sigma_2)_{\alpha\beta}\,\psi^{k}_{VS}(\mathbf{k}) \\ &+(\tau_k\tau_2)^{ab}(\sigma_{k'}\sigma_2)_{\alpha\beta}\,\psi^{kk'}_{VV}(\mathbf{k})\Big].\end{aligned} \tag{2.2}$$

The potential (2.1) reduces for each isospin-spin channel to

$$\begin{aligned}
V_{SS}(\mathbf{k},\mathbf{k}') &= -3\left(\frac{g_A}{2f_\pi}\right)^2\frac{(\mathbf{k}-\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2}+C_S-3C_T\,,\\
V^{i'j'}_{SV}(\mathbf{k},\mathbf{k}') &= 3\left(\frac{g_A}{2f_\pi}\right)^2 \times\frac{(\mathbf{k}-\mathbf{k}')^2\delta^{i'j'}-2(\mathbf{k}-\mathbf{k}')^{i'}(\mathbf{k}-\mathbf{k}')^{j'}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} +(C_S+C_T)\,\delta^{i'j'}\,,\\
V^{ij}_{VS}(\mathbf{k},\mathbf{k}') &= \left(\frac{g_A}{2f_\pi}\right)^2\frac{(\mathbf{k}-\mathbf{k}')^2\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} +(C_S-3C_T)\,\delta^{ij}\,,\\
V^{ij,i'j'}_{VV}(\mathbf{k},\mathbf{k}') &= -\left(\frac{g_A}{2f_\pi}\right)^2 \times\delta^{ij}\frac{(\mathbf{k}-\mathbf{k}')^2\delta^{i'j'}-2(\mathbf{k}-\mathbf{k}')^{i'}(\mathbf{k}-\mathbf{k}')^{j'}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} +(C_S+C_T)\,\delta^{ij}\,\delta^{i'j'}\,.
\end{aligned} \tag{2.3}$$

We still have to implement the Fermi symmetry. This implies that the irreducible wave functions (2.2) must fulfill (isospin and spin indices will be omitted for the rest of this section)

$$\begin{aligned}
\psi_{SS}(\mathbf{k}) &= -\psi_{SS}(-\mathbf{k})\,,\\
\psi_{SV}(\mathbf{k}) &= \psi_{SV}(-\mathbf{k})\,,\\
\psi_{VS}(\mathbf{k}) &= \psi_{VS}(-\mathbf{k})\,,\\
\psi_{VV}(\mathbf{k}) &= -\psi_{VV}(-\mathbf{k})\,,
\end{aligned} \tag{2.4}$$

which is implemented in the LS equation if we choose

$$T_{SS}(\mathbf{k},\mathbf{k}';E) = \frac{1}{2}\left(V_{SS}(\mathbf{k},\mathbf{k}') - V_{SS}(-\mathbf{k},\mathbf{k}')\right) + \frac{1}{2}\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\left(V_{SS}(\mathbf{k},\mathbf{k}'') - V_{SS}(-\mathbf{k},\mathbf{k}'')\right) \times \frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T_{SS}(\mathbf{k}'',\mathbf{k}';E) \qquad (SS \longleftrightarrow VV),$$

$$T_{SV}(\mathbf{k},\mathbf{k}';E) = \frac{1}{2}\left(V_{SV}(\mathbf{k},\mathbf{k}') + V_{SV}(-\mathbf{k},\mathbf{k}')\right) + \frac{1}{2}\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\left(V_{SV}(\mathbf{k},\mathbf{k}'') + V_{SV}(-\mathbf{k},\mathbf{k}'')\right) \times \frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T_{SV}(\mathbf{k}'',\mathbf{k}';E) \qquad (SV \longleftrightarrow VS). \tag{2.5}$$

It is the advantage of the above decomposition that we will not need to specify which (coupled) partial waves we are analyzing.

If the LS equation for the potentials (2.3) was well defined, using (2.5) would be equivalent to solving the LS equation:

$$\widehat{T}_{xy}(\mathbf{k},\mathbf{k}';E) = V_{xy}(\mathbf{k},\mathbf{k}') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V_{xy}(\mathbf{k},\mathbf{k}'') \times \frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\widehat{T}_{xy}(\mathbf{k}'',\mathbf{k}';E), \tag{2.6}$$

($x,y{=}S,V$), namely ignoring the statistics and then using the standard formulas

$$T_{SS}(\mathbf{k},\mathbf{k}';E) = \frac{1}{2}\left(\widehat{T}_{SS}(\mathbf{k},\mathbf{k}';E) - \widehat{T}_{SS}(-\mathbf{k},\mathbf{k}';E)\right) \qquad (SS \to VV),$$
$$T_{SV}(\mathbf{k},\mathbf{k}';E) = \frac{1}{2}\left(\widehat{T}_{SV}(\mathbf{k},\mathbf{k}';E) + \widehat{T}_{SV}(-\mathbf{k},\mathbf{k}';E)\right) \qquad (SV \to VS). \tag{2.7}$$

However, the LS equation for $\widehat{T}_{xy}$ is not well defined in any channel and hence using (2.5) or (2.6)-(2.7) may not be totally equivalent. In particular, for the SS and VV channels, the UV divergences one finds using (2.5) are softer than those from (2.6)-(2.7), so we shall work with (2.5). For the SV and VS channels, however, the UV divergences found using (2.5) are as strong as the ones that stem from (2.6)-(2.7). For convenience, we have chosen to work with the latter for these channels.

The LS equation in the isoscalar-scalar channel in (2.5) is already well defined, as is apparent from the antisymmetrization of the corresponding potential (2.3). On the contrary, the other three channels require regularization. Searching for the systematics to tackle them will be the aim of the next three sections. For notation simplicity, the energy dependence of the T-matrices as well as of other auxiliary functions will not be displayed explicitly for the rest of the paper.

3 The isovector-singlet channel

The LS equation for this channel reads

$$\widehat{T}^{ij}_{VS}(\mathbf{k},\mathbf{k}') = V^{ij}_{VS}(\mathbf{k},\mathbf{k}') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{ik}_{VS}(\mathbf{k},\mathbf{k}'') \times \frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\widehat{T}^{kj}_{VS}(\mathbf{k}'',\mathbf{k}'),$$

where

$$V^{ij}_{VS}(\mathbf{k},\mathbf{k}') = \left\{c_0 + \frac{c_2}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2}\right\}\delta^{ij},$$
$$c_0 := C_S - 3C_T + \left(\frac{g_A}{2f_\pi}\right)^2,$$
$$c_2 := -\left(\frac{g_A m_\pi}{2f_\pi}\right)^2, \tag{3.1}$$

where in the last lines we display the values those constants take if the matching to HBχL is carried out at LO in χPT. However, for the rest of the analysis we need not specify which order in χPT c_0 and c_2 have been calculated at. The hat and the VS subscript will be dropped in the following.

Let us define

$$\mathcal{A}(\mathbf{k}')\,\delta^{ij} := \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{T^{ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}. \tag{3.2}$$

Then (3.1) reads:

$$T^{ij}(\mathbf{k},\mathbf{k}') = c_0(1+\mathcal{A}(\mathbf{k}'))\,\delta^{ij} + \frac{c_2\,\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{c_2}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T^{ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} \tag{3.3}$$

and can be rewritten after solving

$$T_2(\mathbf{k},\mathbf{k}') = \frac{1}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3} \times \frac{c_2}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_2(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}, \tag{3.4}$$

in the form

$$T(\mathbf{k},\mathbf{k}') = c_2\,T_2(\mathbf{k},\mathbf{k}') + c_0(1+\mathcal{A}(\mathbf{k}')) \times \left[1 + c_2\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{T_2(\mathbf{k},\mathbf{k}'')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\right], \tag{3.5}$$

where we have dropped the δ^{ij} structure. If $\mathcal{A}(\mathbf{k}')$ were a fixed function, the equation above would be well defined and could already be solved with no need to regularize it. However, $\mathcal{A}(\mathbf{k}')$ is a functional of T and a second equation which relates them must be introduced. This is achieved

$$c_0(1+\mathcal{A}(\mathbf{k}')) = \frac{1+c_2\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_2(\mathbf{k},\mathbf{k}')}{E-\frac{\mathbf{k}^2}{M}+i\eta}}{\frac{1}{c_0}-\left[\mathcal{I}_0+c_2\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{k}^2}{M}+i\eta}T_2(\mathbf{k},\mathbf{k}'')\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\right]}\,, \qquad \mathcal{I}_0 := \int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{k}^2}{M}+i\eta}\,. \tag{3.6}$$

by multiplying eq. (3.5) by $1/(E-\frac{\mathbf{k}^2}{M}+i\eta)$ and integrating over $\mathbf{k}$. We obtain

see equation (3.6) *above*

Substituting iteratively T_2 in (3.4) in the rhs of (3.6) we see that only the first iteration produces further divergent expressions when $\Lambda\to\infty$. We can then write (3.6) as

$$c_0(1+\mathcal{A}(\mathbf{k}')) = \frac{1+c_2\mathcal{F}(\mathbf{k}')}{\frac{1}{c_0}-[\mathcal{I}_0+c_2\,\mathcal{L}+c_2\mathcal{F}'\,]}\,, \tag{3.7}$$

where $\mathcal{I}_0$ and $\mathcal{L}$ contain linearly and logarithmically divergent terms, respectively, whereas $\mathcal{F}$ ($\mathcal{F}'$) just denote finite functions:

$$\begin{aligned}\mathcal{L} &:= \int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{k}^2}{M}+i\eta}\\ &\quad\times\frac{1}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\,,\\ \mathcal{F}(\mathbf{k}') &:= \int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_2(\mathbf{k},\mathbf{k}')}{E-\frac{\mathbf{k}^2}{M}+i\eta}\,,\\ \mathcal{F}' &:= \int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\int^{\Lambda}\frac{\mathrm{d}^3k'''}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{k}^2}{M}+i\eta}\\ &\quad\times\frac{c_2}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_2(\mathbf{k}'',\mathbf{k}''')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\frac{1}{E-\frac{\mathbf{k}'''^2}{M}+i\eta}\,.\end{aligned} \tag{3.8}$$

It is clear that the expression (3.7) can be renormalized by a redefinition of c_0. In dimensional regularization, ($D=3+2\epsilon$), we obtain

$$\begin{aligned}\frac{1}{c_0} &= -\frac{M^2\,c_2}{4(4\pi)^2}\left(\frac{1}{\epsilon}+\chi_{\mathrm{sch}}\right)+\frac{1}{c_0^r(\mu)}\,,\\ \chi_{MS} &= 0\,,\\ \chi_{\overline{MS}} &= \gamma_E-\mathrm{Log}(4\pi)\,,\end{aligned} \tag{3.9}$$

which is in agreement with [18], and for a hard cut-off:

$$\frac{1}{c_0} = -\frac{M\Lambda}{2\pi^2}+\frac{M^2\,c_2}{32\pi^2}\,\mathrm{Log}\left(\frac{\Lambda^2}{\mu^2}\right)+\frac{1}{c_0^r(\mu)}\,. \tag{3.10}$$

If we now wish to solve numerically the LS equation, we should proceed as usual and introduce a hard cut-off. However, c_0 is not to be fitted to the experimental data but substituted by (3.10) and the cut-off made as large as possible (in practice, it should be enough if $\sqrt{EM}/\Lambda$ is of the order of neglected subleading contributions from the NLO potential —see [21] for a more technical discussion). What we have just proved is that the result will be cut-off independent up to corrections $\sqrt{EM}/\Lambda$. μ must be fixed at the relevant momentum scale $\mu\sim(\sqrt{EM},m_\pi)$ and $c_0^r(\mu)$ tuned to fit the experimental data.

It was already noticed in ref. [18] that if one takes the LO χPT value for c_2, the counterterm c_0 requires a contribution of NLO in χPT. In our approach this only means that the matching calculation in order to get c_0 from the HBχL must be done at least at NLO in χPT.

Although we have no prediction for $c_0^r(\mu)$, we can try to understand from (3.10) how large scattering lengths may arise. Since $c_0^r(\mu)$ evolves according to a non-perturbative renormalization group (RG) equation it might take very different values depending on the scale it is evaluated at. After solving it,

$$c_0^r(\mu) = \frac{c_0^r(\mu_0)}{1+\frac{M^2c_2c_0^r(\mu_0)}{16\pi^2}\mathrm{Log}\frac{\mu}{\mu_0}}\,. \tag{3.11}$$

If we input the value of ref. [18] $c_0^r(m_\pi)=-(\frac{1}{79\ \mathrm{MeV}})^2$, we obtain $c_0^r(M)=-(\frac{1}{125\ \mathrm{MeV}})^2$, which is not quite at the natural scale ($\sim M$). Hence, the non-perturbative low-energy dynamics does not seem to be enough to fill the gap between the natural scales and the large scattering lengths. In spite of that, the variation of $c_0^r(\mu)$ from m_π to M is large enough as to justify a non-perturbative treatment of the OPE in this channel.

4 The isosinglet-vector channel

The LS equation for this channel reads:

$$\begin{aligned}\widehat{T}^{ij}_{SV}(\mathbf{k},\mathbf{k}') &= V^{ij}_{SV}(\mathbf{k},\mathbf{k}')+\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{ik}_{SV}(\mathbf{k},\mathbf{k}'')\\ &\quad\times\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}\widehat{T}^{kj}_{SV}(\mathbf{k}'',\mathbf{k}')\,,\end{aligned}$$

where

$$V_{SV}^{ij}(\mathbf{k},\mathbf{k}') = \left\{ c_0 + \frac{c_2}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} \right\} \delta^{ij}$$

$$+c_1 \frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} ,$$

$$c_0 := C_S + C_T + 3\left(\frac{g_A}{2f_\pi}\right)^2 ,$$

$$c_1 := -6\left(\frac{g_A}{2f_\pi}\right)^2 ,$$

$$c_2 := -3\left(\frac{g_A m_\pi}{2f_\pi}\right)^2 , \tag{4.1}$$

where we display the values of the coupling constants when the matching to HBχPT is carried out at leading order in χPT. Recall again that the analysis which follows holds for arbitrary c_0 , c_1 and c_2 independently of the matching values that these parameters may take at the HBχPT scale. We shall drop the subscript SV and the hat in the following. We call the term proportional to c_1 above spin symmetry breaking (SSB) term. This term breaks the orbital angular momentum conservation and makes the analysis of this channel qualitatively different from the previous one. In order to illustrate this, let us take $\mathbf{k}' = \mathbf{0}$ for simplicity. As we regulate (4.1), the possible divergences arising when the regulator is removed depend on the high-momentum behavior of $T^{ij}(\mathbf{k})$. If $T^{ij}(\mathbf{k}) \sim |\mathbf{k}|^\alpha$, the usual power counting arguments imply that, due to the SSB term, the integral on the rhs will rise this power by one. Hence, the high-momentum behavior of the lhs of the equation will not match the one of its rhs unless: i) $\alpha = -1$ and the high-momentum contribution of the potential cancels out the one arising from the integral or ii) $\alpha = 0$ and the bare coupling constant c_1 goes to zero as the cut-off goes to infinity, which removes the $|\mathbf{k}|^{\alpha+1}$ term on the rhs. We prove in appendix A that the case i) in fact reduces to ii).

The preceding discussion provides a rather intuitive introduction to what, in the course of sect. 4.1, we will demonstrate in full detail. That is, all those rising divergences caused by the SSB term can only be renormalized by a, so far undetermined, flowing of their accompanying coupling constant, c_1. Next, we will fix this cut-off dependence and, having explored the consequences such a behavior has on the amplitude, will come back in sect. 4.2 to standard procedures. There it is shown that the alternative of treating SSB as a perturbation solves the problem, as all divergences get renormalized by local counterterms and no c_1 flowing is longer required.

4.1 Non-perturbative treatment of the SSB term

Let us then return to eq. (4.1). It has the following structure:

$$\begin{aligned} T^{ij}(\mathbf{k},\mathbf{k}') &= c_0(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}')) \\ &\quad +c_1\left[\frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + \mathcal{B}^{ij}(\mathbf{k},\mathbf{k}')\right] \\ &\quad +c_2\frac{\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + c_2\int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3} \\ &\quad \times \frac{1}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T^{ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \\ \mathcal{A}^{ij}(\mathbf{k}') &= \int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T^{ij}(\mathbf{k},\mathbf{k}')}{E-\frac{\mathbf{k}^2}{M}+i\eta} , \\ \mathcal{B}^{ij}(\mathbf{k},\mathbf{k}') &= \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T^{kj}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} . \end{aligned} \tag{4.1.1}$$

Let us define

$$\begin{aligned} T^{ij}(\mathbf{k},\mathbf{k}') &:= c_0(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}'))\,T_0(\mathbf{k}) \\ &\quad +c_1\,T_1^{ij}(\mathbf{k},\mathbf{k}') + c_2\,T_2(\mathbf{k},\mathbf{k}')\,\delta^{ij} , \\ T_0(\mathbf{k}) &= 1 + c_2\int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3} \\ &\quad \times \frac{1}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_0(\mathbf{k}'')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \\ T_1^{ij}(\mathbf{k},\mathbf{k}') &= \frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + \mathcal{B}^{ij}(\mathbf{k},\mathbf{k}') \\ &\quad + c_2\int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_1^{ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \\ T_2(\mathbf{k},\mathbf{k}') &= \frac{1}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + c_2\int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3} \\ &\quad \times \frac{1}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_2(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \end{aligned} \tag{4.1.2}$$

which allows us to isolate in $T_1^{ij}(\mathbf{k},\mathbf{k}')$ and $c_0\,(\delta^{ij} + \mathcal{A}^{ij}(\mathbf{k}'))$ all sources of divergent behavior, since $T_0(\mathbf{k})$ and $T_2(\mathbf{k},\mathbf{k}')$ are perfectly well defined.

Using the expressions of $\mathcal{B}^{ij}(\mathbf{k},\mathbf{k}')$ in (4.1.1) and $T^{ij}(\mathbf{k},\mathbf{k}')$ in (4.1.2), $T_1^{ij}(\mathbf{k},\mathbf{k}')$ can be re-casted in the form

$$\begin{aligned} T_1^{ij}(\mathbf{k},\mathbf{k}') &= c_0(\delta^{kj}+\mathcal{A}^{kj}(\mathbf{k}'))\,T_{10}^{ik}(\mathbf{k}) \\ &\quad +T_{11}^{ij}(\mathbf{k},\mathbf{k}') + c_2 T_{12}^{ij}(\mathbf{k},\mathbf{k}') , \\ T_{10}^{ij}(\mathbf{k}) &= \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^j}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_0(\mathbf{k}'')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} \\ &\quad + \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{c_1(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k + c_2\,\delta^{ik}}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2} \\ &\quad \times \frac{T_{10}^{kj}(\mathbf{k}'')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \end{aligned}$$

$$T_{11}^{ij}(\mathbf{k},\mathbf{k}') = \frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} + \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3} \times \frac{c_1(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k + c_2\,\delta^{ik}}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_{11}^{kj}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta},$$

$$T_{12}^{ij}(\mathbf{k},\mathbf{k}') = \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^j}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_2(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} + \int^\Lambda \frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{c_1(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k + c_2\,\delta^{ik}}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T_{12}^{kj}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}. \tag{4.1.3}$$

This decomposition enables us to compute $c_0(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}'))$, and hence the full amplitude $T^{ij}(\mathbf{k},\mathbf{k}')$, in terms of $T_0(\mathbf{k})$, $T_{1n}^{ij}(\mathbf{k},\mathbf{k}')$ $(n=0,1,2)$ and $T_2(\mathbf{k},\mathbf{k}')$ through the equation

$$\left[\frac{\delta^{ik}}{c_0} - \int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_0(\mathbf{k})\,\delta^{ik}}{E-\frac{\mathbf{k}^2}{M}+i\eta} - c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_{10}^{ik}(\mathbf{k})}{E-\frac{\mathbf{k}^2}{M}+i\eta}\right]c_0(\delta^{kj}+\mathcal{A}^{kj}(\mathbf{k}')) = \delta^{ij} + c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_{11}^{ij}(\mathbf{k},\mathbf{k}')+c_2T_{12}^{ij}(\mathbf{k},\mathbf{k}')}{E-\frac{\mathbf{k}^2}{M}+i\eta} + c_2\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_2(\mathbf{k},\mathbf{k}')\,\delta^{ij}}{E-\frac{\mathbf{k}^2}{M}+i\eta}. \tag{4.1.4}$$

As we have already mentioned, $T_0(\mathbf{k})$ and $T_2(\mathbf{k},\mathbf{k}')$ are finite when the cut-off is removed. If we solve $T_{1n}^{ij}(\mathbf{k},\mathbf{k}')$, $n=0,1,2$ iteratively, the most divergent pieces in the n-th iteration are $T_{10}\sim(c_1\Lambda)^n\Lambda$, $T_{11}\sim(c_1\Lambda)^n$ and $T_{12}\sim(c_1\Lambda)^{n-1}c_1$. These series are expected to have a finite radius of convergence. The radius of convergence is in any case non-zero because they are bounded by geometric series (or derivatives of them). If c_1 does not go to zero as $1/\Lambda$ or stronger (in particular, if c_1 is not allowed to flow), each series will separately diverge. In that case, a finite result can only be obtained if non-trivial cancellations occur for all n, which we do not see how they could actually happen. If, on the contrary,

$$c_1(\Lambda) = \frac{\bar{c}_1}{\Lambda} + \dots, \tag{4.1.5}$$

and $\bar{c}_1$ is small enough, the series will converge. For the T-matrix, such a strong cut-off dependence implies that the terms

$$c_1T_{10}^{ij}(\mathbf{k}) \longrightarrow t_{10}^{(0)}\delta^{ij} + \frac{t_{10}^{ij}(\mathbf{k})}{\Lambda} + \dots,$$
$$c_1T_{11}^{ij}(\mathbf{k},\mathbf{k}') \longrightarrow \frac{t_{11}^{ij}(\mathbf{k},\mathbf{k}')}{\Lambda} + \dots,$$
$$c_1T_{12}^{ij}(\mathbf{k},\mathbf{k}') \longrightarrow \frac{t_{12}^{ij}(\mathbf{k},\mathbf{k}')}{\Lambda} + \dots, \tag{4.1.6}$$

where $t_{10}^{(0)}$ is simply a finite constant and, as we can see, all $(\mathbf{k},\,\mathbf{k}')$-encoded information will be washed out from the amplitude.

That is to say:

$$T^{ij}(\mathbf{k},\mathbf{k}') = \lim_{\Lambda\to\infty} c_0(\delta^{kj}+\mathcal{A}^{kj}(\mathbf{k}'))\left(T_0(\mathbf{k})\delta^{ik}+c_1T_{10}^{ik}(\mathbf{k})\right) + c_1T_{11}^{ij}(\mathbf{k},\mathbf{k}') + c_2\left(T_2(\mathbf{k},\mathbf{k}')\,\delta^{ij} + c_1T_{12}^{ij}(\mathbf{k},\mathbf{k}')\right) = c_0(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}'))\left(T_0(\mathbf{k})+c_1\,t_{10}^{(0)}\right) + c_2T_2(\mathbf{k},\mathbf{k}')\delta^{ij}, \tag{4.1.7}$$

which is finite provided $c_0\,(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}'))$ is finite. In order to prove the latter, we borrow from sect. 3 the following results:

$$\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_0(\mathbf{k})}{E-\frac{k^2}{M}+i\eta} = -\frac{M\Lambda}{2\pi^2} + \frac{M^2c_2}{32\pi^2}\mathrm{Log}\left(\frac{\Lambda^2}{\mu^2}\right) + \mathcal{O}(1),$$
$$\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_2(\mathbf{k},\mathbf{k}')}{E-\frac{k^2}{M}+i\eta} = \mathcal{O}(1), \tag{4.1.8}$$

and find in appendix B:

$$c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_{10}^{ii}(\mathbf{k})}{E-\frac{k^2}{M}+i\eta} = a_0\Lambda + ib_0\sqrt{EM} + d_0\mathrm{Log}\left(\frac{\Lambda}{m_\pi}\right) + \mathcal{O}\left(\frac{1}{\Lambda}\right),$$
$$c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_{11}^{ii}(\mathbf{k},\mathbf{k}')}{E-\frac{k^2}{M}+i\eta} = \mathcal{O}(1),$$
$$c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_{12}^{ii}(\mathbf{k},\mathbf{k}')}{E-\frac{k^2}{M}+i\eta} = \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{4.1.9}$$

where a_0, b_0, d_0 are cut-off independent constants related to $\bar{c}_1$. Then the flow

$$\frac{1}{c_0} = -\frac{M\Lambda}{2\pi^2} + \frac{a_0\Lambda}{3} + \frac{M^2c_2}{32\pi^2}\mathrm{Log}\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{d_0}{6}\mathrm{Log}\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{1}{c_0^r(\mu)} \tag{4.1.10}$$

makes $c_0(\delta^{ij}+\mathcal{A}^{ij}(\mathbf{k}'))$ finite and hence (4.1.7) does. We have then proved that the flows (4.1.5) and (4.1.10) renormalize the triplet channel.

It is not difficult to see that the various series above involving divergent terms are bounded by geometric series or derivatives of them. This ensures that our flows provide actually finite expressions for the amplitude if $\bar{c}_1$ is small enough. However, this amplitude appears to be diagonal in spin space and hence orbital angular momentum is conserved. Although, the observed $^3S_1$-$^3D_1$ mixing, which is small, might be attributed to a higher-order effect, it is clear from ref. [25] that it is due to the OPE to a large extent. In order to preclude the conservation of orbital angular momentum, we can foresee two ways out: i) a SSB term may survive in the renormalized amplitude if $\bar{c}_1$ is tuned infinitely close to the radius of convergence of the series, so that our bounds do not hold anymore, and ii) the

SSB term from OPE must be treated as a perturbation and renormalized as such. The possibility i) is examined in appendix C, where we show it unlikely to be realized. In the following subsection we explore ii) and prove that if a suitable SSB term is treated as a perturbation, the amplitude is renormalizable at first order and the mixing survives.

4.2 Treating the SSB term perturbatively

Let us split the potential as

$$\begin{aligned} V^{ij}(\mathbf{k},\mathbf{k}') &= V^{(0)\,ij}(\mathbf{k},\mathbf{k}') + V^{(1)\,ij}(\mathbf{k},\mathbf{k}'),\\ V^{(0)\,ij}(\mathbf{k},\mathbf{k}') &= \left\{\widetilde{c}_0 + \frac{\widetilde{c}_2}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2}\right\}\delta^{ij},\\ V^{(1)\,ij}(\mathbf{k},\mathbf{k}') &= \widetilde{c}_1\frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j - \frac{(\mathbf{k}-\mathbf{k}')^2}{3}\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2},\\ \widetilde{c}_0 &:= C_S + C_T + \left(\frac{g_A}{2f_\pi}\right)^2,\\ \widetilde{c}_1 &:= -6\left(\frac{g_A}{2f_\pi}\right)^2,\\ \widetilde{c}_2 &:= -\left(\frac{g_A m_\pi}{2f_\pi}\right)^2, \end{aligned} \tag{4.2.1}$$

with LO values for the coupling constants indicated. In the following we drop the SV-channel sub-indices.

The amplitude will be written as

$$T^{ij}(\mathbf{k},\mathbf{k}') = T^{(0)\,ij}(\mathbf{k},\mathbf{k}') + T^{(1)\,ij}(\mathbf{k},\mathbf{k}'), \tag{4.2.2}$$

where $T^{(0)\,ij}(\mathbf{k},\mathbf{k}')$ fulfills

$$T^{(0)\,ij}(\mathbf{k},\mathbf{k}') = V^{(0)\,ij}(\mathbf{k},\mathbf{k}') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{(0)\,ik}(\mathbf{k},\mathbf{k}'') \times\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T^{(0)\,kj}(\mathbf{k}'',\mathbf{k}'). \tag{4.2.3}$$

The renormalized solution to this equation is given by $T^{(0)\,ij}(\mathbf{k},\mathbf{k}') = T(\mathbf{k},\mathbf{k}')\delta^{ij}$ in sect. 3. At first order in perturbation theory $T^{(1)\,ij}(\mathbf{k},\mathbf{k}')$ verifies

$$\begin{aligned} T^{(1)\,ij}(\mathbf{k},\mathbf{k}') &= V^{(1)\,ij}(\mathbf{k},\mathbf{k}')\\ &+\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{(1)\,ik}(\mathbf{k},\mathbf{k}'')\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T^{(0)\,kj}(\mathbf{k}'',\mathbf{k}')\\ &+\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{(0)\,ik}(\mathbf{k},\mathbf{k}'')\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T^{(1)\,kj}(\mathbf{k}'',\mathbf{k}'). \end{aligned} \tag{4.2.4}$$

Using (3.4) and (3.5) we can see that the second term above is finite. We can then gather the first and second terms into a new, energy-dependent, potential defined as

$$\widetilde{V}^{(1)\,ij}(\mathbf{k},\mathbf{k}'') := V^{(1)\,ij}(\mathbf{k},\mathbf{k}') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V^{(1)\,ik}(\mathbf{k},\mathbf{k}'') \times\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T^{(0)\,kj}(\mathbf{k}'',\mathbf{k}'). \tag{4.2.5}$$

Therefore, the integral equation reduces to

$$\begin{aligned} T^{(1)\,ij}(\mathbf{k},\mathbf{k}') &= \widetilde{V}^{(1)\,ij}(\mathbf{k},\mathbf{k}'') + \widetilde{c}_0\,\mathcal{R}^{ij}(\mathbf{k}')\\ &+\int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{\widetilde{c}_2}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{T^{(1)\,ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta},\\ \mathcal{R}^{ij}(\mathbf{k}') &:= \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}\frac{T^{(1)\,ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}. \end{aligned} \tag{4.2.6}$$

In order to prove it finite we decompose

$$T^{(1)\,ij}(\mathbf{k},\mathbf{k}') = \widetilde{c}_0\,\mathcal{R}^{kj}(\mathbf{k}')\,T_0^{ik}(\mathbf{k}) + \widetilde{T}_1^{ij}(\mathbf{k},\mathbf{k}'), \tag{4.2.7}$$

with $T_0^{ij}(\mathbf{k})$ defined in (4.1.2) and $\widetilde{T}_1^{ij}(\mathbf{k},\mathbf{k}')$ given by

$$\widetilde{T}_1^{ij}(\mathbf{k},\mathbf{k}') := \widetilde{V}^{(1)\,ij}(\mathbf{k},\mathbf{k}'') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3} \times\frac{\widetilde{c}_2}{(\mathbf{k}-\mathbf{k}'')^2+m_\pi^2}\frac{\widetilde{T}_1^{ij}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta}. \tag{4.2.8}$$

Both $T_0^{ij}(\mathbf{k})$ and $\widetilde{T}_1^{ij}(\mathbf{k},\mathbf{k}')$ are well defined (the tensor structure is crucial for the latter to be so). Divergences can only arise in $\widetilde{c}_0\,\mathcal{R}^{ij}(\mathbf{k}')$, which reads

$$\widetilde{c}_0\,\mathcal{R}^{ij}(\mathbf{k}') = \frac{\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\widetilde{T}_1^{ij}(\mathbf{k},\mathbf{k}')}{E-\frac{\mathbf{k}^2}{M}+i\eta}}{\widetilde{c}_0^{-1}-\frac{1}{3}\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T_0^{ii}(\mathbf{k})}{E-\frac{\mathbf{k}^2}{M}+i\eta}}. \tag{4.2.9}$$

The numerator is well defined (for that the tensor structure is again crucial) and the divergences in the denominator have exactly the same structure as in the denominator of (3.7). Hence they are renormalized by the same c_0 flows. We have then proved that if we treat the SSB term as a perturbation, the amplitude is renormalizable at first order in perturbation theory and no extra counterterm needs to be introduced.

5 Isovector-vector channel

If we use (2.6)-(2.7) in order to obtain $T_{VV}(\mathbf{k},\mathbf{k}')$, the calculation of $\widehat{T}_{VV}(\mathbf{k},\mathbf{k}')$ would reduce to that of the previous section. However, as mentioned in sect. 2, the UV behavior is smoother in terms of (2.5), as happens in the SS-channel, although here we still need to introduce a regularization. The LS equation, dropping the isospin delta, reads:

$$T_{VV}^{ij}(\mathbf{k},\mathbf{k}') = V_{VV}^{A,ij}(\mathbf{k},\mathbf{k}') + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3}V_{VV}^{A,ik}(\mathbf{k},\mathbf{k}'') \times\frac{1}{E-\frac{\mathbf{k}''^2}{M}+i\eta}T_{VV}^{kj}(\mathbf{k}'',\mathbf{k}'), \tag{5.1}$$

where

$$\begin{aligned} V_{VV}^{A,ij}(\mathbf{k},\mathbf{k}') &= \frac{1}{2}\left(V_{VV}^{ij}(\mathbf{k},\mathbf{k}') - V_{VV}^{ij}(-\mathbf{k},\mathbf{k}')\right) =\\ &\frac{c_1}{2}\left(\frac{(\mathbf{k}-\mathbf{k}')^i(\mathbf{k}-\mathbf{k}')^j}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} - \frac{(\mathbf{k}+\mathbf{k}')^i(\mathbf{k}+\mathbf{k}')^j}{(\mathbf{k}+\mathbf{k}')^2+m_\pi^2}\right)\\ &+\frac{c_2}{2}\left(\frac{\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} - \frac{\delta^{ij}}{(\mathbf{k}+\mathbf{k}')^2+m_\pi^2}\right), \end{aligned} \tag{5.2}$$

where those constants calculated at first order in χPT take the values

$$c_1 := 2\left(\frac{g_A}{2f_\pi}\right)^2 ,\qquad c_2 := \left(\frac{g_A m_\pi}{2f_\pi}\right)^2 . \tag{5.3}$$

We have not analyzed the possible existence of non-trivial flows which may renormalize the above equation. The fact that the SSB term must be treated perturbatively in the SV-channel, indicates that also here we should proceed according to the same philosophy. The potential (5.2) in the zeroth order approximation reads:

$$V_{VV}^{(0),\,ij}(\mathbf{k},\mathbf{k}') = \frac{c_2}{2}\left(\frac{\delta^{ij}}{(\mathbf{k}-\mathbf{k}')^2+m_\pi^2} - \frac{\delta^{ij}}{(\mathbf{k}+\mathbf{k}')^2+m_\pi^2}\right), \tag{5.4}$$

which leads to a well-defined LS equation. At first order in perturbation theory we will have

$$\begin{aligned} T_{VV}^{ij}(\mathbf{k},\mathbf{k}') &= T_{VV}^{(0)\,ij}(\mathbf{k},\mathbf{k}') + T_{VV}^{(1)\,ij}(\mathbf{k},\mathbf{k}') ,\\ T_{VV}^{(1)\,ij}(\mathbf{k},\mathbf{k}') &= V_{VV}^{(1)\,ij}(\mathbf{k},\mathbf{k}') \\ &\quad + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3} V_{VV}^{(1)\,ik}(\mathbf{k},\mathbf{k}'')\frac{T_{VV}^{(0)\,kj}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} \\ &\quad + \int^{\Lambda}\frac{\mathrm{d}^3k''}{(2\pi)^3} V_{VV}^{(0)\,ik}(\mathbf{k},\mathbf{k}'')\frac{T_{VV}^{(1)\,kj}(\mathbf{k}'',\mathbf{k}')}{E-\frac{\mathbf{k}''^2}{M}+i\eta} , \end{aligned} \tag{5.5}$$

which is also well defined. We expect the divergences arising at higher orders to be absorbed by local counterterms.

6 Discussion

We have addressed the renormalization of the LS equation for the LO potentials (in the χPT counting) of the NN system in all channels. In addition, for each channel we have been able to carry out our analysis for all partial waves (including partial-wave mixing) at once. The isoscalar-scalar channel does not require regularization. For the isovector-scalar channel we recover the flows of ref. [18]. The remaining two channels have deserved a more detailed study.

The first non-trivial result is that the renormalization of the isoscalar-vector channel requires a strong flow of the coupling constant of a non-local potential, the SSB one, or, in other words, if only the coupling constants of the local potentials are allowed to flow, the isoscalar-vector channel is not renormalizable. Several comments are in order.

First of all, the flow (4.1.5) of the coupling constant of the SSB term is not such a big surprise. Notice that at high momentum this term tends to a (direction-dependent) constant, which is the same behavior (except for the direction dependence) as the δ-function term both in the singlet and the triplet channel, the coupling constants of which also show similar flows. The main difference is that the leading behavior for c_0 is fixed and the subleading one contains the free parameter ($c_0^r(\mu)$). For c_1 instead, the leading behavior contains the free parameter ($\bar{c}_1$) and the subleading behavior is not observable.

What is worse, the flow (4.1.5) has undesirable consequences: the renormalized T-matrix conserves orbital angular momentum, even if the bare interaction does not (see appendix C)[4]. Since it is precisely the OPE the main responsible for mixing (also of higher partial waves [25]), we would like it to keep doing this job for us. Therefore, in order for c_1 not to flow, but to be fixed at the HBχL values and produce partial-wave mixing, we are forced to exclude the SSB term from the (low-energy) LO potential, and to treat it as a perturbation. This also appears to be reasonable from the phenomenological point of view, since the observed mixings are small [25].

We have developed this line in sects. 4.2 and 5. We have proved that at first order the vector channels remain renormalizable (at zeroth order the problem reduces to the one in the singlet channels, which are renormalizable). The picture which emerges is half-way between [11], where the pions are treated perturbatively, and [5,17], where the whole potential is treated non-perturbatively. The (low-energy) LO potential has the form of the LO potential in the χPT counting which conserves orbital angular momentum. We are tempted to propose the following counting. The $\mathcal{O}(Q^n)$ ($n = 0, 1, ...$) contribution to the NN potential must be divided into two pieces: the one which conserves orbital angular momentum (SS) and the one which does not (SSB). The SSB terms keep their leading χPT counting but the SS ones are enhanced and must be counted as $\mathcal{O}(Q^{n-1})$. Only the LO potential $\mathcal{O}(Q^{-1})$ must be treated (and renormalized) non-perturbatively. We have seen here that this proposal is theoretically consistent at next-to-leading order, and, in addition, it does not require any coupling constant of a non-local potential to flow anymore. Notice that the difference with respect to the expansion of ref. [11] consists in including a piece of the one-pion exchange in the (low-energy) LO potential. This piece vanishes in the UV, which makes us believe that the renormalization properties of the theory will be similar to those in the expansion of ref. [11]. However, the convergence properties, which are also sensible to the IR, will be different, and hopefully better. Whether the latter is so or not requires a N^{2}LO calculation to be compared with that of ref. [12], which is beyond the scope of this paper.

Let us finally comment on recent work on the subject [19,20]. The authors in both references try to renormalize the triplet channel by adjusting the coupling constant of the δ-potential only. Hence, according to our

[4] We have also checked perturbatively in $\bar{c}_1$ and c_2 up to order $\bar{c}_1 c_2$ that the effective range depends on $\bar{c}_1$ only through the scattering length. Since the latter can be adjusted by tuning $c_0^r(\mu)$, up to this order both the scattering length and the effective range are blind to $\bar{c}_1$. We have not looked at what happens to the rest of the amplitude or to higher orders but we suspect that they are also insensitive to $\bar{c}_1$.

results, both works should show a remnant cut-off dependence when the cut-off is large enough. Note also that it is only in the large cut-off limit when a meaningful comparison is possible, since the regularizations used in the three works are different. The authors of ref. [20], who use a subtracted (μ-dependent) LS equation, argue that a reasonable boundary condition is that for large μ the T-matrix coincides with the potential, and check numerically whether, once the scattering lengths are fixed, the remaining observables are independent of μ for large μ. They find that for laboratory energies up to 100 MeV the ${}^3S_1$ and ${}^3D_1$ phase shifts are remarkably independent of μ for $\mu \geq 0.8$ GeV, but the mixing angle shows a strong μ-dependence for $6\,\text{GeV} \geq \mu \geq 0.8\,\text{GeV}$ and only for $\mu \geq 6$ GeV the μ-dependence smooths and the results may appear to converge. We interpret this stronger μ-dependence of the mixing angle as an indication of the remnant cut-off dependence mentioned above. The authors of ref. [19] obtain the flows by analyzing the short-distance behavior of the Schrödinger equation (see also [26]). For the ${}^1S_0$ they are in qualitative agreement with ours. For the triplet channel they present analytic flow equations which are argued to coincide with those of the chiral limit. The flow of the δ-function term is given implicitly by their eq. (18). They assume that their α_π, which is proportional to our c_1, does not flow[5] and find a multi-branch structure for the flow of their V_0R^3, which is proportional to our c_0 ($R \to 0$, R playing the role of an inverse cut-off). It is interesting to note that if they allowed α_π flow like our c_1 in subsect. 4.1, namely $\alpha_\pi \sim R$, and V_0R^3 like our c_0, namely $V_0 \sim 1/R^2$, their eq. (18) becomes cut-off independent. Hence our flow (4.1.5) is a solution in the $R \to 0$ limit to the flow equation (18) of ref. [19]. Recall, however, that, if α_π is not allowed to flow, the strict limit $R \to 0$ cannot be taken. This is proved in appendix D. Hence eq. (18) of [19] does not produce an acceptable flow for V_0 and, therefore, it cannot be used to properly renormalize the triplet channel in the $R \to 0$ limit. This is consistent with the fact that we did not find any solution in sect. 4.1 (when c_1 was not allowed to flow), since we were only analyzing the large cut-off ($R \to 0$) limit. Moreover, as explained in ref. [19] (see also [26]), the claim of the renormalizability there is to be understood as follows: R (the cut-off) is kept finite, but the effect due to the finiteness of R is shown to be a higher-order effect in the EFT expansion. Whereas, at first sight, this may appear to be a reasonable procedure within an EFT framework, the flows of eq. (18) of [19] could eventually lead to problems. If we wish to improve the accuracy of our EFT calculation, we will have to calculate at higher orders. Even if we insist in keeping R finite, we will have to choose it smaller and smaller for the LO terms not to jeopardize the accuracy of the higher-order calculation. Then at some point R will hit the region where no continuous solution exists and we will loose all predictive power (if we give up continuity, an infinite number of inequivalent solutions exists). Note that the fact that finite cut-off effects can be compensated by higher-dimensional operators [27], which holds in perturbatively renormalizable (and asymptotically free) theories, needs not hold here. One should admit, however, that this might actually happen at very high orders, so that the procedure proposed in ref. [19] may prove useful in practice.

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Appendix A. The case $\alpha = -1$

The more general decomposition of the high momentum behavior of $T^{ij}(\mathbf{k})$ for $\alpha = -1$ reads:

$$T^{ij}(\mathbf{k}) = \mathcal{B}_{-1}\frac{\delta^{ij}}{|\mathbf{k}|} + \widetilde{\mathcal{B}}_{-1}\frac{\mathbf{k}^i\mathbf{k}^j}{|\mathbf{k}|^3} + \mathcal{P}^{ij}(\mathbf{k})\,, \tag{A.1}$$

where $\lim_{\mathbf{k}\to\infty}\mathcal{P}^{ij}(\mathbf{k}) \sim \frac{1}{\mathbf{k}^2}$. Notice then that the integral in (4.1) at most diverges logarithmically and, furthermore, the divergent term must be proportional to the δ^{ij} tensor. By calculating the high-energy behavior of the integral in the rhs of (4.1) we obtain

$$\begin{aligned} T^{ij}(\mathbf{k}) \sim\; & c_0\delta^{ij} + c_1\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^2} - \frac{Mc_0}{2\pi^2}\left[\mathcal{B}_{-1} + \frac{\widetilde{\mathcal{B}}_{-1}}{3}\right] \\ & \times \mathrm{Log}\left(\frac{\Lambda^2}{-EM}\right)\delta^{ij} + \frac{Mc_1}{4\pi^2}\frac{\mathcal{B}_{-1}+\widetilde{\mathcal{B}}_{-1}}{3} \\ & \times\left[\mathrm{Log}\left(\frac{\mathbf{k}^2}{\Lambda^2}\right) + f_1\right]\delta^{ij} - \frac{Mc_1}{4\pi^2}\left[\mathcal{B}_{-1} + \frac{\widetilde{\mathcal{B}}_{-1}}{3}\right] \\ & \times\left[\mathrm{Log}\left(\frac{\mathbf{k}^2}{-EM}\right) + f_2\right]\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^2}\,, \end{aligned} \tag{A.2}$$

with f_1 and f_2 two finite, constant terms. Observe that, although the cut-off dependence can be removed by a suitable redefinition of c_0, the non-analytic terms $\sim \mathrm{Log}|\mathbf{k}|$ cannot be compensated by the potential. Self-consistency of (A.1) and (A.2) force $c_1 \to 0$ again.

Appendix B. Proof of (4.1.9)

Let us define

$$\mathcal{H}_\alpha(EM) := c_1\int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{T^{ii}_{1\alpha}(\mathbf{k})}{E - \frac{\mathbf{k}^2}{M} + i\eta}, \qquad \alpha = 0,1,2\,, \tag{B.1}$$

[5] Whereas the combination $\alpha_\pi m_\pi^2$ that appears in the singlet channel is equivalent to out c_2 and, accordingly, remains fixed.

and concentrate on $\mathcal{H}_0(EM)$ (the analysis for $\mathcal{H}_1(EM)$ is identical). We have

$$\mathcal{H}_0(EM) = \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \int^{\Lambda} \frac{d^3k''}{(2\pi)^3} \frac{c_1}{E - \frac{\mathbf{k}^2}{M} + i\eta} \times \frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^j}{(\mathbf{k}-\mathbf{k}'')^2 + m_\pi^2} \frac{T_0(\mathbf{k}'')}{E - \frac{\mathbf{k}^2}{M} + i\eta} + \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \int^{\Lambda} \frac{d^3k''}{(2\pi)^3} \frac{c_1}{E - \frac{\mathbf{k}^2}{M} + i\eta} \times \frac{c_1(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k + c_2\delta^{ik}}{(\mathbf{k}-\mathbf{k}'')^2 + m_\pi^2} \frac{T_{10}^{kj}(\mathbf{k}'')}{E - \frac{\mathbf{k}^2}{M} + i\eta}. \quad \text{(B.2)}$$

If we solve the equation above iteratively using (4.1.2) for $T_0(\mathbf{k})$ and (4.1.3) for $T_{10}^{ij}(\mathbf{k})$, the most divergent term in the n-th iteration is (super-indices j and k are contracted with unwritten momenta):

$$c_1^{n+1} \left\{ \prod_{l=1}^{n+2} \int^{\Lambda} \frac{d^3k_l}{(2\pi)^3} \right\} \frac{1}{E - \frac{\mathbf{k}_1^2}{M} + i\eta} \times \frac{(\mathbf{k}_1-\mathbf{k}_2)^i(\mathbf{k}_1-\mathbf{k}_2)^j}{(\mathbf{k}_1-\mathbf{k}_2)^2 + m_\pi^2} \frac{1}{E - \frac{\mathbf{k}_2^2}{M} + i\eta} \cdots \cdots \frac{1}{E - \frac{\mathbf{k}_{n+1}^2}{M} + i\eta} \frac{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^k(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^i}{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^2 + m_\pi^2} \times \frac{1}{E - \frac{\mathbf{k}_{n+2}^2}{M} + i\eta}. \quad \text{(B.3)}$$

Taking into account that the limits $E \to 0$ and $m_\pi^2 \to 0$ exist and the flow (4.1.5), the leading behavior in Λ reads:

$$(-M)^{n+2} c_1^{n+1} \left\{ \prod_{l=1}^{n+2} \int^{\Lambda} \frac{d^3k_l}{(2\pi)^3} \frac{1}{\mathbf{k}_l^2} \right\} \frac{(\mathbf{k}_1-\mathbf{k}_2)^i(\mathbf{k}_1-\mathbf{k}_2)^j}{(\mathbf{k}_1-\mathbf{k}_2)^2} \cdots \cdots \frac{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^k(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^i}{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^2} \sim \bar{c}_1^{n+1}\Lambda, \quad \text{(B.4)}$$

which proves that a_0 is a ($\bar{c}_1$-dependent) constant. Notice also that the integral in (B.4) is bound by $(\int^{\Lambda} d^3\mathbf{k}/\mathbf{k}^2)^{n+2}$. Let us next identify the subleading behavior. Consider first $E = 0$. The derivative of (B.3) with respect to m_π^2 at $m_\pi^2 = 0$ has at most a logarithmic singularity which means that the next-to-leading behavior in Λ is $\sim c_1^{n+1}\Lambda^{n-1}m_\pi^2 \mathrm{Log}\Lambda$, which gives rise to $\mathcal{O}\left(\frac{1}{\Lambda}\right)$ contributions in (4.1.9). Terms contributing to d_0 in the n-th iteration appear when: i) the c_2-proportional term of $T_0(\mathbf{k})$ is iterated through only c_1 potential insertions coming from the second line in (B.2); ii) the equal-to-1 term of $T_0(\mathbf{k})$ is iterated in such a way that a c_2 potential from the last piece appears only once in the iteration. The relevant integral is obtained by substituting

$$c_1 \frac{(\mathbf{k}_p-\mathbf{k}_{p+1})^i(\mathbf{k}_p-\mathbf{k}_{p+1})^j}{(\mathbf{k}_p-\mathbf{k}_{p+1})^2 + m_\pi^2} \longrightarrow \frac{c_2\,\delta^{ij}}{(\mathbf{k}_p-\mathbf{k}_{p+1})^2 + m_\pi^2} \quad \text{(B.5)}$$

in (B.3). In order to get the leading behavior in Λ of this integral we can set $m_\pi^2 = 0$ in all but the substituted term above. We have (super-indices j, l, q and k are contracted with unwritten momenta):

$$(-M)^{n+2} c_2\, c_1^n \left\{ \prod_{l=1\setminus\{p,p+1\}}^{n} \int^{\Lambda} \frac{d^3k_l}{(2\pi)^3} \frac{1}{\mathbf{k}_l^2} \right\} \times \frac{(\mathbf{k}_1-\mathbf{k}_2)^i(\mathbf{k}_1-\mathbf{k}_2)^j}{(\mathbf{k}_1-\mathbf{k}_2)^2} \cdots \left[\int^{\Lambda} \frac{d^3k_p}{(2\pi)^3} \int^{\Lambda} \frac{d^3k_{p+1}}{(2\pi)^3} \frac{(\mathbf{k}_{p-1}-\mathbf{k}_p)^l}{(\mathbf{k}_{p-1}-\mathbf{k}_p)^2} \frac{1}{\mathbf{k}_p^2} \times \frac{(\mathbf{k}_{p-1}-\mathbf{k}_p)\cdot(\mathbf{k}_{p+1}-\mathbf{k}_{p+2})}{(\mathbf{k}_p-\mathbf{k}_{p+1})^2 + m_\pi^2} \frac{1}{\mathbf{k}_{p+1}^2} \frac{(\mathbf{k}_{p+1}-\mathbf{k}_{p+2})^q}{(\mathbf{k}_{p+1}-\mathbf{k}_{p+2})^2} \right] \cdots \frac{(\mathbf{k}_{n-1}-\mathbf{k}_n)^k(\mathbf{k}_{n-1}-\mathbf{k}_n)^i}{(\mathbf{k}_{n-1}-\mathbf{k}_n)^2} \sim c_2\,\bar{c}_1^n\,\mathrm{Log}\Lambda, \quad \text{(B.6)}$$

which proves (with the flow (4.1.1)) that d_0 is a constant.

Let us next address the energy-dependent contribution to (B.1). Notice that any analytic contribution in EM would show up at $\mathcal{O}(1/\Lambda)$. Hence only non-analytic contributions (like the one in (B.6)) are relevant to us. Let us then look for non-analytic contributions in EM in the most divergent diagram in the n-th iteration (B.3). Since the $m_\pi^2 \to 0$ limit exists we can take it and have

$$c_1^{n+1} \int^{\Lambda} \frac{dk_1}{(2\pi)^3} \frac{\mathbf{k}_1^2}{E - \frac{\mathbf{k}_1^2}{M} + i\eta} \cdots \int^{\Lambda} \frac{dk_{n+2}}{(2\pi)^3} \frac{\mathbf{k}_{n+2}^2}{E - \frac{\mathbf{k}_{n+2}^2}{M} + i\eta} \int d\Omega_1 \cdots \int d\Omega_{n+2} \frac{(\mathbf{k}_1-\mathbf{k}_2)^i(\mathbf{k}_1-\mathbf{k}_2)^j}{(\mathbf{k}_1-\mathbf{k}_2)^2} \cdots \cdots \frac{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^k(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^i}{(\mathbf{k}_{n+1}-\mathbf{k}_{n+2})^2}, \quad \text{(B.7)}$$

where $d\Omega_i$, $i = 1, ..., n+2$ stand for angular integrals. Since the most singular contribution comes from the region $|\mathbf{k}_l| \sim \Lambda\ \forall\, l$, the angular integral will give rise to a constant (which, furthermore, is bound by $(4\pi)^{n+2}$), and the integrals over $|\mathbf{k}_l|$ decouple. Hence the leading behavior for small E turns out to be the non-analytic contribution we are looking for (α_0, β_0, $\widetilde{\alpha}_0$ and $\widetilde{\beta}_0$ are constants):

$$\sim c_1^{n+1} \left[\int^{\Lambda} dk \frac{\mathbf{k}^2}{E - \frac{\mathbf{k}^2}{M} + i\eta} \right]^{n+2} \sim c_1^{n+1} \left(\alpha_0\Lambda + i\beta_0\sqrt{EM} + \mathcal{O}\left(\frac{1}{\Lambda}\right) \right)^{n+2} \sim \bar{c}_1^{n+1} \left(\widetilde{\alpha}_0\Lambda + i\widetilde{\beta}_0\sqrt{EM} + \mathcal{O}\left(\frac{1}{\Lambda}\right) \right), \quad \text{(B.8)}$$

which proves, in addition, that b_0 is a constant. Notice that a $\mathrm{Log}\Lambda$-dependence in this term would have been fatal for renormalization.

We have then proved the first formula in (4.1.9). The proof of the second formula is identical. The third formula is proved by simply noticing that all integrals involved are at most logarithmically divergent and, those which actually are, go multiplied by $c_1 \sim \frac{1}{\Lambda}$.

Appendix C. On c_1 tuning

In sect. 4, when we focused on proving that a certain behavior of the bare constants of the potential as functions of the cut-off (namely, c_0, $c_1 \sim \Lambda^{-1}$) would render a finite T-matrix, only c_0 was conveniently fine-tuned. As a result, the so-computed scattering amplitude lacked from partial-wave mixing, which is expected due to the second range tensorial term in the (bare) Hamiltonian. In order to obtain partial-wave mixing, two possibilities must be regarded. On the one hand, it could well happen that, indeed, mixing should not have been considered as LO, but as a NLO term to be treated perturbatively, the divergences it may cause being absorbed in the usual way by higher-order local counterterms. This appears to be consistent with the fact that partial-wave mixing in this channel amounts only to a few degrees. This treatment resums the δ^{ij}-proportional part of OPE. Its SSB term, now eliminated by the strong suppression of c_1, is then recovered in a NLO analysis. We have shown how this works in the sect. 4.2.

Nevertheless, another possibility remains unexamined. A proper tuning of c_1 to a, let us say, non-trivial RG fixed point, could very well recover mixing at the leading order. So far, the existence of such a fixed point is anything but evident. Uncovering it or ruling it out requires detailed numerical work which is beyond the scope of this paper. However, in order to illustrate our point, let us provide two approximations that exemplify how this tuning would emerge, how it would affect previous results and to which extent to achieve this goal we depend on the exact resolution of our actual system of integral equations.

Let us take in the following $\mathbf{k}' = \mathbf{0}$ for simplicity. We will also apply the chiral limit (m_π, $c_2 \to 0$) and work with $\widetilde{c}_0$ and $\widetilde{c}_1$ defined in sect. 4.2 . After decomposing the T-matrix in

$$T^{ij}(\mathbf{k}) = T_1(k)\,\delta^{ij} + \left[\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2}\right] T_2(k)\,, \qquad \text{(C.1)}$$

the following two angular integrals arise in the resolution of its LS equation:

$$\widetilde{c}_1 \int \frac{\mathrm{d}\Omega''}{4\pi}\; \frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^j - \frac{(\mathbf{k}-\mathbf{k}'')^2}{3}\delta^{ij}}{(\mathbf{k}-\mathbf{k}'')^2} \longrightarrow \widetilde{c}_1\,\omega_1\left(\frac{k}{k''}\right)\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2}\,,$$

$$\begin{aligned}\widetilde{c}_1 \int \frac{\mathrm{d}\Omega''}{4\pi}\, &\frac{(\mathbf{k}-\mathbf{k}'')^i(\mathbf{k}-\mathbf{k}'')^k - \frac{(\mathbf{k}-\mathbf{k}'')^2}{3}\delta^{ik}}{(\mathbf{k}-\mathbf{k}'')^2} \\ &\times\left[\frac{\mathbf{k}''^k\mathbf{k}''^j - \frac{\mathbf{k}''^2}{3}\delta^{kj}}{\mathbf{k}''^2}\right] \\ &\longrightarrow \left[\widetilde{c}_1\,\omega_2\left(\frac{k}{k''}\right)\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2} + \widetilde{c}_1\,\omega_3\left(\frac{k}{k''}\right)\delta^{ij}\right], \end{aligned} \qquad \text{(C.2)}$$

with $\omega_i\left(\frac{k}{k''}\right)$, $i = 1,2,3$, as known functions ($k = |\mathbf{k}|$, $k'' = |\mathbf{k}''|$).

At this point we wish to introduce some reasonable approximation that allows us to transform the non-separable in k and k'' functions $\omega_i\left(\frac{k}{k''}\right)$ into separable ones. Once this is achieved, we only need to solve a conventional system of equations and check whether, at least within this approximation, a non-trivial fixed point exists. Obviously, our approximation should be as compatible as possible with what we know about the behavior of the full $\mathrm{d}^3\mathbf{k}$-integrals. For instance:

$$\int^{\Lambda} \frac{\mathrm{d}k''}{2\pi^2}\frac{k''^2\,\omega_i\left(\frac{k}{k''}\right)}{E - \frac{k''^2}{M} + i\eta} \sim k\,, \qquad i = 1,2\,, \qquad \text{(C.3)}$$

that is, both are finite integrals proportional to k in the limit $\Lambda \to \infty$. Unfortunately, no separable ω_i achieves this. We shall content ourselves with a simple but still reasonable starting point that enforces separability. Then, let us take $\omega_3\left(\frac{k}{k''}\right)$ as a constant ($:= \alpha_3$) and substitute $\omega_{1,2}\left(\frac{k}{k''}\right)$ by $:= \alpha_{1,2}\,\frac{k}{k''}$ ($\alpha_{1,2}$ also being constants). Although the latter introduces logarithmic divergences which do not exist in the actual function, it keeps the correct behavior in k shown in (C.3).

The LS equation takes the form

$$\begin{aligned} T^{ij}(\mathbf{k}) = {}& \widetilde{c}_0(1 + T_1\,\mathcal{I}_0)\,\delta^{ij} + \widetilde{c}_1\left[\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2}\right] \\ &+\widetilde{c}_1\alpha_1\left[\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2}\right] k\,T_1\,\mathcal{I}_{-1} \\ &+\widetilde{c}_1\alpha_2\left[\frac{\mathbf{k}^i\mathbf{k}^j - \frac{\mathbf{k}^2}{3}\delta^{ij}}{\mathbf{k}^2}\right] k\,\mathcal{C} + \widetilde{c}_1\,\alpha_3\,\delta^{ij}\,\mathcal{B}\,, \end{aligned} \qquad \text{(C.4)}$$

where we have already used that $T_1(k)$ becomes momentum independent, as is easily verified through (C.4):

$$\begin{aligned} T_1 &= \widetilde{c}_0(1 + T_1\,\mathcal{I}_0) + \widetilde{c}_1\,\alpha_3\,\mathcal{B}\,, \\ T_2(k) &= \widetilde{c}_1(1 + \alpha_1\,k\,T_1\,\mathcal{I}_{-1} + \alpha_2\,k\,\mathcal{C})\,. \end{aligned} \qquad \text{(C.5)}$$

The following functions have been introduced:

$$\mathcal{I}_0 := \int^{\Lambda} \frac{\mathrm{d}k}{2\pi^2} \frac{k^2}{E - \frac{k^2}{M} + i\eta},$$
$$\mathcal{B} := \int^{\Lambda} \frac{\mathrm{d}k}{2\pi^2} \frac{k^2\, T_2(k)}{E - \frac{k^2}{M} + i\eta},$$
$$\mathcal{I}_{-1} = \int^{\Lambda} \frac{\mathrm{d}k}{2\pi^2} \frac{k}{E - \frac{k^2}{M} + i\eta},$$
$$\mathcal{C} = \int^{\Lambda} \frac{\mathrm{d}k}{2\pi^2} \frac{k\, T_2(k)}{E - \frac{k^2}{M} + i\eta}. \quad \text{(C.6)}$$

A few manipulations allow us to solve for T_1 and the combination $\alpha_1\, T_1\, \mathcal{I}_{-1} + \alpha_2\, \mathcal{C}$:

$$T_1 = \frac{\widetilde{c}_0 + \widetilde{c}_1^2 \alpha_3 \mathcal{I}_0 + \widetilde{c}_1^3 \alpha_1 \alpha_3 \frac{\mathcal{I}_1 \mathcal{I}_{-1}}{1-\widetilde{c}_1\alpha_2\mathcal{I}_0}}{1 - \widetilde{c}_0 \mathcal{I}_0 - \widetilde{c}_1^2 \alpha_1 \alpha_3 \frac{\mathcal{I}_1 \mathcal{I}_{-1}}{1-\widetilde{c}_1\alpha_2\mathcal{I}_0}},$$
$$\alpha_1\, T_1 \mathcal{I}_{-1} + \alpha_2\, \mathcal{C} = \left(\frac{\widetilde{c}_1 \alpha_2 \mathcal{I}_{-1}}{1 - \widetilde{c}_1 \alpha_2 \mathcal{I}_0}\right) \times \frac{\left(\widetilde{c}_0 + \widetilde{c}_1^2 \alpha_3 \mathcal{I}_0\right) \frac{\alpha_1}{\alpha_2} + \widetilde{c}_1 (1 - \widetilde{c}_0 \mathcal{I}_0)}{1 - \widetilde{c}_0 \mathcal{I}_0 - \widetilde{c}_1^2 \alpha_1 \alpha_3 \frac{\mathcal{I}_1 \mathcal{I}_{-1}}{1-\widetilde{c}_1\alpha_2\mathcal{I}_0}}, \quad \text{(C.7)}$$

where a quadratic divergence

$$\mathcal{I}_1 := \int^{\Lambda} \frac{\mathrm{d}k}{2\pi^2} \frac{k^3}{E - \frac{k^2}{M} + i\eta} \quad \text{(C.8)}$$

enters.

It is not difficult to realize that little has been gained: the only way to get (C.7) finite is by an untuned $\widetilde{c}_1$, $(1 - \widetilde{c}_1 \alpha_2 \mathcal{I}_0 \neq 0)$, and a tuned $\widetilde{c}_0$, which force $T_2(k)$ to become trivial again. We have not been able to figure out any reasonable approximation which produces a non-trivial $T_2(k)$.

Anyway, in order to illustrate the kind of fixed point we are looking for, let us take another option which, unfortunately, is completely unrealistic. It consists in sending $\omega_1\left(\frac{k}{k''}\right)$ and $\omega_3\left(\frac{k}{k''}\right)$ to zero, keeping $\omega_2\left(\frac{k}{k''}\right)$ as a mere constant $(:= \alpha_2)$. This presents the main advantage of producing decoupled equations for T_1 and T_2:

$$T_1 = \widetilde{c}_0 (1 + \mathcal{A}),$$
$$T_2 = \widetilde{c}_1 (1 + \alpha_2\, \mathcal{B}), \quad \text{(C.9)}$$

where $\mathcal{A} := T_1\, \mathcal{I}_0$ and $\mathcal{B} = T_2\, \mathcal{I}_0$. Both are well defined, provided $\widetilde{c}_0(1+\mathcal{A})$ and $\widetilde{c}_1(1+\alpha_2\, \mathcal{B})$ are finite. We compute them multiplying above by $1/(E - \frac{\mathbf{k}^2}{M} + i\eta)$ and integrating. This produces

$$\widetilde{c}_0 (1 + \mathcal{A}) = \frac{1}{\frac{1}{\widetilde{c}_0} - \mathcal{I}_0},$$
$$\widetilde{c}_1 (1 + \alpha_2\, \mathcal{B}) = \frac{1}{\frac{1}{\widetilde{c}_1} - \alpha_2\, \mathcal{I}_0}. \quad \text{(C.10)}$$

It is obvious that divergences are absorbed if $\widetilde{c}_0$, $\widetilde{c}_1$ behave like Λ^{-1} and non-trivial results $(T_1, T_2 \neq 0)$ require

$$\frac{1}{\widetilde{c}_0} := -\frac{M\Lambda}{2\pi^2} + \frac{1}{\widetilde{c}_0^r(\mu)},$$
$$\frac{1}{\widetilde{c}_1} := -\frac{M\Lambda\alpha_2}{2\pi^2} + \frac{1}{\widetilde{c}_1^r(\mu)}. \quad \text{(C.11)}$$

Namely, $\widetilde{c}_1$ must be fine-tuned (to a non-trivial fixed point) as desired. Unfortunately, as mentioned before, the assumptions made for the ω_i here are not realistic.

Summarizing, we are rather pessimistic about the possibility that a non-trivial RG fixed point for both $\widetilde{c}_0$ and $\widetilde{c}_1$ exists, which allows for partial-wave mixing at leading order.

Appendix D. Proof that no continuous solution of eq. (18) of [19] exists when R → 0

Consider eq. (18) of [19],

$$\sqrt{-MV_0}R \cot(\sqrt{-MV_0}R) = \frac{3}{4} + \sqrt{\frac{6M\alpha_\pi}{R}} \tan\left(2\sqrt{\frac{6M\alpha_\pi}{R}} + \phi_0\right), \quad \text{(D.1)}$$

with $V_0 < 0$, $R > 0$, $\phi_0 \in [-\pi/2, \pi/2]$. We are interested in whether continuous solutions $V_0 = V_0(R)$ exist when $R \to 0$. Let us define

$$y := \sqrt{-MV_0}R > 0, \qquad x := 2\sqrt{\frac{6M\alpha_\pi}{R}} + \phi_0. \quad \text{(D.2)}$$

In terms of these variables we are interested in whether a continuous solution $y = y(x)$ exists when $x \to \infty$ for the following equation:

$$y \cot y = \frac{3}{4} + \frac{x - \phi_0}{2} \tan x. \quad \text{(D.3)}$$

Deriving this equation once one obtains

$$\frac{\sin 2y - 2y}{2\sin^2 y} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin 2x + 2x - 2\phi_0}{4\cos^2 x}, \quad \text{(D.4)}$$

which proves that $y(x)$ decreases when x increases for x large enough. The proof holds everywhere except for the points $x = (n + 1/2)\pi$, $y = m\pi$, $n, m = 0, 1, 2...$, which we analyze in the following.

When x approaches $(n + 1/2)\pi$ for a given n, y must necessarily approach $m\pi$ for some m in order for eq. (D.4) to have a solution. If we write

$$x = \left(n + \frac{1}{2}\right)\pi + \delta x, \quad y = m\pi + \delta y, \quad \delta x, \ \delta y \to 0, \quad \text{(D.5)}$$

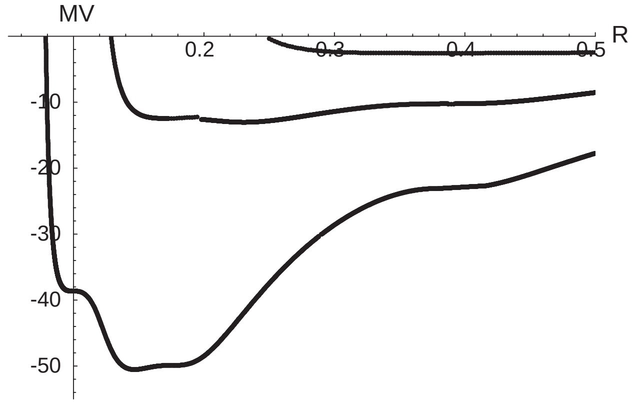

Fig. 1. In the figure it is shown how the first three branches of the flow presented in fig. 4 of [19] behave as one approaches the relevant limit $R \to 0$.

we have, for $m \neq 0$,

$$\delta y = -\frac{2m}{n + \frac{1}{2} - \frac{\phi_0}{\pi}} \delta x + \mathcal{O}(\delta x^2) . \tag{D.6}$$

Hence, eq. (D.4) admits a continuous solution near the point $x = (n + 1/2)\pi$, provided that we choose $m \neq 0$. Notice also that y keeps decreasing when x increases in the neighborhood of this point.

Now, if we increase x from $(n + 1/2)\pi$ to $(n + 3/2)\pi$, y must decrease from $m\pi$ to $(m - 1)\pi$, if continuity is required. By iterating the argument, if we increase x till $(n + m + 1/2)\pi$, continuity requires y to decrease till 0. However, for $x = (n + m + 1/2)\pi + \delta x$ ($\delta x \to 0$) and $y = \delta y \to 0$, eq. (D.4) does not have a solution anymore, since one obtains

$$1 + \mathcal{O}(\delta y^2) = -\frac{(n + m + \frac{1}{2})\pi - \phi_0}{2\delta x} + \mathcal{O}(1) . \tag{D.7}$$

This implies, in particular, that the curves plotted in fig. 4 of [19] cannot be continuously extended below $R \sim 0.25\,\mathrm{fm}$, $R \sim 0.13\,\mathrm{fm}$ and $R \sim 0.09\,\mathrm{fm}$, respectively, as is shown in fig. 1.

In conclusion, no continuous solution $y = y(x)$ of eq. (D.4) (and hence of eq. (D.1)) exists for $x \to \infty$ ($R \to 0$). If continuity is given up, an infinite number of solutions exist, none of them being compatible with a RG flow, at least in the standard sense. Note also that this situation is qualitatively different from a limit cycle behavior [28], which is realized, for instance, in three-body systems [29]. There the flows are oscillating and discontinuous but uniquely defined, no matter how large the cut-off is. It is interesting to notice, however, that for any fixed R arbitrarily small there are always branches for which R has an image (by choosing m above sufficiently large).

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